

The instability spectrum of weakly-magnetized SU(2) Reissner-Nordström black holes

Shahar Hod

The Ruppin Academic Center, Emek Hefer 40250, Israel
and

The Hadassah Institute, Jerusalem 91010, Israel

(Dated: October 29, 2014)

It is well known that the U(1) Reissner-Nordström black hole is stable within the framework of the Einstein-Maxwell theory. However, the SU(2) Reissner-Nordström black-hole solution of the coupled Einstein-Yang-Mills equations is known to be unstable. In fact, this magnetically charged black hole is characterized by an infinite set of unstable (growing in time) perturbation modes. In the present paper we study *analytically* the instability resonance spectrum of weakly-magnetized SU(2) Reissner-Nordström black holes. In particular, we obtain explicit analytical expressions for the infinite set $\{\omega_n\}_{n=0}^{\infty}$ of imaginary eigenvalues that characterize the instability growth rates of the perturbation modes. We discuss the role played by these unstable eigenvalues as critical exponents in the gravitational collapse of the Yang-Mills field. Finally, it is shown that our analytical formulas for the characteristic black-hole instability spectrum agree with new numerical data that recently appeared in the literature.

I. INTRODUCTION

The Einstein-Yang-Mills theory has attracted much attention from both physicists and mathematicians since the discovery, made by Bartnik and McKinnon [1], of a discrete family of regular self-gravitating solitonic solutions of the coupled equations. As shown by Bizoń [2] (see also [3]), the theory also admits a discrete family of hairy black-hole solutions, known as colored black holes [4]. In fact, it was shown by Yasskin [5] already in the 70s that the coupled Einstein-Yang-Mills equations admit an explicit solution in the form of a magnetically charged Reissner-Nordström black hole.

These solutions of the coupled Einstein-Yang-Mills equations are known to be *unstable* [6–8]. In particular, the n th colored black-hole solution is characterized by n unstable (growing in time) perturbation modes [9]. As for the magnetized SU(2) Reissner-Nordström black hole, it was proved in [8] that this unstable solution of the coupled Einstein-Yang-Mills equations is characterized by an *infinite* set of unstable perturbation modes. This fact is quite surprising since the U(1) Reissner-Nordström black hole is known to be stable within the framework of the coupled Einstein-Maxwell theory [10] (see also [11]).

So why is it interesting to study these unstable solutions of the Einstein-Yang-Mills theory? One important reason lies in the fact that these unstable configurations have been identified as *critical* solutions [12] of the coupled Einstein-Yang-Mills equations [13–16]. That is, these unstable configurations play the role of intermediate attractors in the dynamical gravitational collapse of the Yang-Mills field [17].

In particular, it has been demonstrated numerically [13–16] that, during a near-critical gravitational collapse of the Yang-Mills field, the time spent in the vicinity of the critical solution (that is, the time spent in the vicinity of an unstable black-hole configuration of the Einstein-Yang-Mills theory) exhibits a critical scaling behavior of

the form [18]

$$\tau = \text{const} - \gamma \ln |p - p^*|, \quad (1)$$

where the critical exponents are directly related to the instability eigenvalues that characterize the relevant unstable black-hole (critical) solution [13–16]:

$$\gamma = 1/\omega_{\text{instability}}. \quad (2)$$

It is therefore of physical interest to study the instability spectra which characterize the black-hole solutions of the coupled Einstein-Yang-Mills equations. A detailed numerical study of the instability spectrum of the $n = 1$ colored black holes can be found in [15]. Most recently, Rinne [16] has computed numerically the instability eigenvalues which characterize the SU(2) Reissner-Nordström black holes in the framework of the Einstein-Yang-Mills theory [19].

The main goal of the present paper is to determine *analytically* the instability spectrum (that is, the *infinite* set of imaginary eigenvalues) which characterizes the SU(2) Reissner-Nordström black-hole spacetime. As we shall show below, the Schrödinger-like wave equation [see Eq. (6) below] which governs the dynamics of linear perturbations to the SU(2) Reissner-Nordström black-hole spacetime is amenable to an analytical treatment in the regime of weakly-magnetized black holes [20].

II. DESCRIPTION OF THE SYSTEM

The SU(2) Reissner-Nordström black-hole solution with unit magnetic charge is described by the line element [5]

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (3)$$

where the mass function $m = m(r)$ is given by [21]

$$m(r) = M - \frac{1}{2r}. \quad (4)$$

The black-hole outer horizon is located at

$$r_+ = M + \sqrt{M^2 - 1} . \quad (5)$$

The dynamics of linearized perturbations $\xi(r)e^{-i\omega t}$ of the black-hole spacetime is governed by the Schrödinger-like wave equation [22]

$$\left[\frac{d^2}{dx^2} + \omega^2 - U(x) \right] \xi = 0 , \quad (6)$$

where the “tortoise” radial coordinate x is defined by [23]

$$dx/dr = [1 - 2m(r)/r]^{-1} , \quad (7)$$

and the effective binding potential in (6) is given by

$$U[x(r)] = -\frac{1}{r^2} \left[1 - \frac{2m(r)}{r} \right] . \quad (8)$$

Note that unstable (growing in time) modes are characterized by

$$\Im \omega > 0 . \quad (9)$$

These unstable modes may equivalently be regarded as ‘bound states’ (characterized by $\omega^2 < 0$) of the effective binding potential (8).

III. THE LARGE-MASS LIMIT

In this paper we shall consider weakly-magnetized black holes whose unit magnetic charge is small on the scale set by the black-hole mass:

$$M \gg 1 . \quad (10)$$

The Schrödinger-like wave equation (6) for these large (weakly-magnetized) black holes can be approximated by

$$\left[\frac{d^2}{dx^2} + \omega^2 + \left(1 - \frac{2M}{r} \right) \frac{1}{r^2} \right] \xi = 0 . \quad (11)$$

The effective potential in (11) is negative in the entire range $-\infty < x < \infty$ and it vanishes asymptotically for $x \rightarrow \pm\infty$. As noted in [22], this fact guarantees that the Schrödinger-like wave equation (11) possesses at least one *unstable* mode with $\omega^2 < 0$ (that is, at least one bound state with negative energy [24]). In fact, as we shall show below, the Schrödinger-like wave equation (11) is characterized by an *infinite* set $\{\omega_n\}_{n=0}^{\infty}$ of unstable modes (an infinite set of bound-state resonances) with $\Im \omega > 0$.

As we shall now show, the wave equation (11) is amenable to an analytical treatment in the regime (10) of large black-hole masses (that is, in the regime of weakly-magnetized black holes). We first point out that the Schrödinger-like equation (11) is of the same form as the familiar Regge-Wheeler equation [25]

$$\left[\frac{d^2}{dx^2} + \omega^2 - \left(1 - \frac{2M}{r} \right) \frac{l(l+1)}{r^2} \right] \xi = 0 \quad (12)$$

which describes electromagnetic perturbations of frequency ω and angular harmonic index l in the Schwarzschild black-hole spacetime. In our case, the effective harmonic index acquires a complex value [compare Eqs. (11) and (12)] [26]:

$$\ell \equiv l_{\text{eff}} = \frac{-1 + i\sqrt{3}}{2} . \quad (13)$$

IV. THE FUNDAMENTAL INSTABILITY EIGENVALUE

In order to calculate the fundamental instability eigenvalue ω_0 of the system, we shall closely follow the analysis of Dolan and Ottewill [27] who provided an elegant method for the calculation of the fundamental black-hole *quasinormal frequencies*. In this section we shall demonstrate that this analytical method can also be applied successfully to the analysis of fundamental *bound-state energies* (in our case, for the calculation of the fundamental instability eigenvalue) [28].

The analytical approach of [27] is based on an expansion of the resonances in inverse powers of the harmonic parameter $L \equiv l + 1/2$:

$$M\omega_{ln} = \sum_{k=-1}^{\infty} w_k L^{-k} , \quad (14)$$

where the expansion coefficients $\{w_k\}_{k=-1}^{k=4}$ are given by equations (17)-(22) of [27]. Substituting

$$L = \ell + \frac{1}{2} = i\frac{\sqrt{3}}{2} \quad (15)$$

into (14), one finds [29]

$$M\omega_0^{\text{ana}} = i \times \frac{413545392 - 108984521\sqrt{3}}{1836660096} \simeq 0.1224i \quad (16)$$

for the fundamental [30] instability eigenvalue of the black hole.

For comparison, the numerically computed fundamental eigenvalue in the large-mass limit (10) is given by [16, 31]

$$M\omega_0^{\text{num}} = 0.1243i . \quad (17)$$

One therefore finds a fairly good agreement (to better than 2%),

$$\frac{\omega_0^{\text{ana}}}{\omega_0^{\text{num}}} \simeq 0.985 , \quad (18)$$

between the *analytically* calculated fundamental eigenvalue (16) and the corresponding *numerically* computed [16] eigenvalue (17).

As discussed in [27], the validity of the expansion method (14) is restricted to the fundamental $n \lesssim l$ modes. In the next section we shall develop a different analytical approach in order to explore the rest of the (infinitely large) family of unstable resonances $\{\omega_n\}_{n=1}^{\infty}$ [32].

V. THE INFINITE SPECTRUM OF UNSTABLE BOUND-STATE RESONANCES

Taking cognizance of (16), one realizes that the entire instability spectrum of the weakly-magnetized black holes is characterized by the relation $M|\omega| < 1$ [33]. In fact, the numerical results of Rinne [16] indicate that the excited eigenvalues $\{\omega_n\}_{n=1}^{\infty}$ of the system are characterized by the property

$$M|\omega_n| \ll 1 \quad ; \quad n = 1, 2, 3, \dots \quad (19)$$

As we shall now show, a *low-frequency* analysis of the perturbation modes can yield, with a remarkably good accuracy, the excited eigenvalues $\{\omega_n\}_{n=1}^{\infty}$ of the unstable black hole.

As shown in [34], the Regge-Wheeler equation (12) is amenable to an analytical treatment in the low-frequency regime (19). In particular, the absorption and reflection coefficients of scattered low-frequency electromagnetic waves in a spherically-symmetric black-hole spacetime were calculated in [34]. The analytical method used in [34] can be summarized as follows: (1) find approximate solutions of Eq. (12) in three spatially distinct regions of the black-hole exterior region, and then (2) use *two* matching procedures (which are based on continuity conditions) in order to overlap the three analytical solutions (see [34] for details).

While the analytical technique used in [34] for the analysis of the low-frequency *scattering* problem can also be applied to the analysis of the *bound-state* resonances of Eq. (11), here we shall use a somewhat simpler trick which involves a *single* (rather than a *double* [34]) matching procedure.

The trick is to analyze the *physically equivalent* Teukolsky radial equation [35]:

$$\Delta^2 \frac{d^2 \psi}{dr^2} + \left[\omega^2 r^4 + 2iM\omega r^2 - \Delta[2i\omega r + \ell(\ell+1)] \right] \psi = 0, \quad (20)$$

which, like Eq. (12), describes the dynamics of electromagnetic perturbation fields in the non-rotating black-hole spacetime [36]. Here $\Delta \equiv r^2 - 2Mr$ and in our case ℓ is given by Eq. (13). It was first proved by Chandrasekhar [37] that the Teukolsky radial equation (20) for non-rotating black holes (also known as the Bardeen-Press equation [38]) is physically equivalent to the Regge-Wheeler equation (12).

As we shall now show, one can derive analytically the entire low frequency instability spectrum $\{\omega_n\}_{n=1}^{\infty}$ of the black hole from Eq. (20) using a single matching procedure [instead of the double matching procedure required for the analysis of Eq. (12)] [39, 40]. We shall look for bound-state ($\omega^2 < 0$) solutions which are characterized by

$$\psi(x \rightarrow -\infty) \sim e^{|\omega|x} \rightarrow 0, \quad (21)$$

and

$$\psi(x \rightarrow \infty) \sim x e^{-|\omega|x} \rightarrow 0, \quad (22)$$

where $\omega = i|\omega|$.

It proves useful to define new dimensionless variables [40, 41]

$$z \equiv \frac{r - 2M}{2M} \quad ; \quad k \equiv -2iM\omega, \quad (23)$$

in terms of which the wave equation (20) becomes

$$z^2(z+1)^2 \frac{d^2 \psi}{dz^2} + \left[-k^2 z^4 + 2kz^3 - \ell(\ell+1)z(z+1) - k(2z+1) - k^2 \right] \psi = 0. \quad (24)$$

The solution of the radial equation (24) in the near-horizon region $kz \ll 1$ which satisfies the boundary condition (21) is given by [40, 41]

$$\psi(z) = z^{1+k}(z+1)^{1-k} {}_2F_1(-\ell+1, \ell+2; 2+2k; -z), \quad (25)$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric function [42].

The solution of the radial equation (24) in the far-region $z \gg 1$ is given by [40, 41]

$$\psi(z) = A e^{kz} z^{\ell+1} {}_1F_1(\ell+2; 2\ell+2; -2kz) + B e^{kz} z^{-\ell} {}_1F_1(-\ell+1; -2\ell; -2kz), \quad (26)$$

where ${}_1F_1(a; c; z)$ is the confluent hypergeometric function [42] and the coefficients $\{A, B\}$ are constants. These coefficients can be determined by matching the two solutions for the radial function, (25) and (26), in the overlap region [43]

$$1 \ll z \ll 1/k. \quad (27)$$

This matching procedure yields [40, 41]

$$A = \frac{\Gamma(2\ell+1)\Gamma(2+2k)}{\Gamma(\ell+2)\Gamma(\ell+1+2k)}, \quad (28)$$

and

$$B = \frac{\Gamma(-2\ell-1)\Gamma(2+2k)}{\Gamma(-\ell+1)\Gamma(-\ell+2k)}. \quad (29)$$

Finally, substituting (28) and (29) into the far-region solution (26) and using the asymptotic ($z \gg 1$) form of the confluent hypergeometric functions [42], one finds [40, 41]

$$\psi(z \rightarrow \infty) = \psi_1 r e^{-kz} + \psi_2 r^{-1} e^{kz}, \quad (30)$$

where

$$\begin{aligned} \psi_1 &= \frac{(2\ell+1)\Gamma^2(2\ell+1)\Gamma(2+2k)}{2\Gamma^2(\ell+2)\Gamma(\ell+1+2k)} (-2k)^{-\ell} M^{-1} \\ &\quad - \frac{(2\ell+1)\Gamma^2(-2\ell-1)\Gamma(2+2k)}{2\Gamma^2(-\ell+1)\Gamma(-\ell+2k)} (-2k)^{\ell+1} M^{-1}, \end{aligned} \quad (31)$$

and

$$\psi_2 = \frac{2(2\ell+1)\Gamma^2(2\ell+1)\Gamma(2+2k)}{\ell(\ell+1)\Gamma^2(\ell)\Gamma(\ell+1+2k)}(2k)^{-\ell-2}M - \frac{2(2\ell+1)(\ell+1)\Gamma^2(-2\ell-1)\Gamma(2+2k)}{\ell\Gamma^2(-\ell)\Gamma(-\ell+2k)}(2k)^{\ell-1}M. \quad (32)$$

A spatially bounded solution which respects the boundary condition (22) is characterized by $\psi(z \rightarrow \infty) \rightarrow 0$. The coefficient ψ_2 in (30) should therefore vanish, which yields the resonance condition [see Eq. (32)]

$$(2k)^{2\ell+1} = \left[\frac{\Gamma(2\ell+1)\Gamma(-\ell)}{(\ell+1)\Gamma(-2\ell-1)\Gamma(\ell)} \right]^2 \frac{\Gamma(-\ell+2k)}{\Gamma(\ell+1+2k)} \quad (33)$$

for the bound-state energies (unstable eigenvalues) of the system. Substituting into (33) the value $\ell = \frac{-1+i\sqrt{3}}{2}$ for the effective harmonic index ℓ [see Eq. (13)] and using Eq. 6.1.18 of [42], one can write the resonance condition (33) in the form

$$k^{i\sqrt{3}} = 8^{i\sqrt{3}} e^{i2\pi/3} \frac{\Gamma^2(i\frac{\sqrt{3}}{2})}{\Gamma^2(-i\frac{\sqrt{3}}{2})} \frac{\Gamma(\frac{1-i\sqrt{3}}{2}+2k)}{\Gamma(\frac{1+i\sqrt{3}}{2}+2k)}. \quad (34)$$

Since k is a small quantity [see Eqs. (19) and (23)], one can use an iteration scheme in order to solve the resonance condition (34). The zeroth-order resonance equation is given by

$$k^{i\sqrt{3}} = 8^{i\sqrt{3}} e^{i2\pi/3} \frac{\Gamma^2(i\frac{\sqrt{3}}{2})}{\Gamma^2(-i\frac{\sqrt{3}}{2})} \frac{\Gamma(\frac{1-i\sqrt{3}}{2})}{\Gamma(\frac{1+i\sqrt{3}}{2})}. \quad (35)$$

Denoting

$$\theta = \arg[\Gamma(i\sqrt{3}/2)] \quad ; \quad \phi = \arg[\Gamma(\frac{1+i\sqrt{3}}{2})], \quad (36)$$

one finds from (35) the *infinite* set

$$M\omega_n^{(0)} = i \times 4e^{-\frac{2\pi}{\sqrt{3}}(n-\frac{1}{3})+\frac{4\theta-2\phi}{\sqrt{3}}} \quad ; \quad n = 1, 2, 3, \dots \quad (37)$$

of zeroth-order unstable ($\Im\omega > 0$) eigenvalues [44].

Substituting (37) into the r.h.s of (34), one obtains the first-order resonance condition

$$k^{i\sqrt{3}} = 8^{i\sqrt{3}} e^{-i2\pi(n-\frac{1}{3})} \times \frac{\Gamma^2(i\frac{\sqrt{3}}{2})}{\Gamma^2(-i\frac{\sqrt{3}}{2})} \frac{\Gamma(\frac{1-i\sqrt{3}}{2}+16e^{-\frac{2\pi}{\sqrt{3}}(n-\frac{1}{3})+\frac{4\theta-2\phi}{\sqrt{3}}})}{\Gamma(\frac{1+i\sqrt{3}}{2}+16e^{-\frac{2\pi}{\sqrt{3}}(n-\frac{1}{3})+\frac{4\theta-2\phi}{\sqrt{3}}})}. \quad (38)$$

Denoting

$$\phi_n = \arg \left[\Gamma(\frac{1+i\sqrt{3}}{2} + 16e^{-\frac{2\pi}{\sqrt{3}}(n-\frac{1}{3})+\frac{4\theta-2\phi}{\sqrt{3}}}) \right], \quad (39)$$

one finds from (38) the infinite family

$$M\omega_n^{(1)} = i \times 4e^{-\frac{2\pi}{\sqrt{3}}(n-\frac{1}{3})+\frac{4\theta-2\phi_n}{\sqrt{3}}} \quad ; \quad n = 1, 2, 3, \dots \quad (40)$$

of first-order unstable eigenvalues [45].

For the first two ‘excited’ eigenvalues one finds from (40)

$$M\omega_1^{(1)\text{ana}} = 1.25 \times 10^{-2}i \quad ; \quad M\omega_2^{(1)\text{ana}} = 3.65 \times 10^{-4}i. \quad (41)$$

For comparison, the corresponding numerically computed eigenvalues in the large-mass limit (10) are given by [16, 46]

$$M\omega_1^{\text{num}} = 1.23 \times 10^{-2}i \quad ; \quad M\omega_2^{\text{num}} = 3.57 \times 10^{-4}i. \quad (42)$$

One therefore finds a fairly good agreement (to within $\sim 2\%$),

$$\frac{\omega_1^{(1)\text{ana}}}{\omega_1^{\text{num}}} \simeq 1.018 \quad ; \quad \frac{\omega_2^{(1)\text{ana}}}{\omega_2^{\text{num}}} \simeq 1.023, \quad (43)$$

between the analytical formula (40) for the unstable eigenvalues of the black hole and the numerically computed [16] eigenvalues (42) [47].

VI. SUMMARY AND PHYSICAL IMPLICATIONS

In summary, we have analyzed the instability spectrum of weakly magnetized SU(2) Reissner-Nordström black holes. In particular, we have derived analytical expressions for the infinite family of unstable (imaginary) black-hole resonances.

The main results derived in this paper and their physical implications are as follows:

(1) For the analysis of the fundamental instability eigenvalue, ω_0 , we have used an expansion method which originally was developed for the analysis of black-hole quasinormal resonances [27]. Here we have demonstrated that this analytical method can also be applied successfully to the analysis of fundamental bound-state energies. The black-hole fundamental instability eigenvalue ω_0 is given by Eq. (16).

(2) For the analysis of the infinitely large spectrum of ‘excited’ eigenvalues [48], $\{\omega_n\}_{n=1}^{n=\infty}$, we have used an appropriate small frequency $M\omega \ll 1$ matching procedure in order to solve the Schrödinger-like wave equation (11) which governs the dynamics of perturbations in the SU(2) Reissner-Nordström black-hole spacetime. The excited instability spectrum $\{\omega_n\}_{n=1}^{n=\infty}$ is given by the analytical formula (40) [47].

(3) We have shown that the *analytically* derived formulas for the characteristic instability spectrum of the weakly magnetized SU(2) Reissner-Nordström black hole, Eqs. (16) and (40), agree with direct *numerical* computations [16] of the eigenvalues.

(4) The interesting numerical work of Rinne [16] has recently revealed that unstable SU(2) Reissner-Nordström black holes may play the role of *approximate*

[49] codimension-two intermediate attractors (critical solutions) in dynamical gravitational collapse of the Yang-Mills field. In particular, it was found [13–16] that, during a near-critical gravitational collapse of the Yang-Mills field, the time spent in the vicinity of the critical solution (that is, the time spent in the vicinity of an unstable black-hole solution of the Einstein-Yang-Mills equations) exhibits a critical scaling law [see Eqs. (1) and (2)], where the critical exponents are given by the reciprocals of the corresponding instability eigenvalues.

Our formulas (16) and (40) provide explicit analytical expressions for these critical exponents (instability eigenvalues) in the regime where the weakly magnetized SU(2) Reissner-Nordström black hole plays the role of

the critical intermediate attractor [16]. To the best of our knowledge, this is the first time that a critical exponent of nontrivial gravitational collapse is calculated analytically.

ACKNOWLEDGMENTS

This research is supported by the Carmel Science Foundation. I would like to thank Oliver Rinne for sharing with me his numerical data. I would also like to thank Yael Oren, Arbel M. Ongo, and Ayelet B. Lata for helpful discussions.

-
- [1] R. Bartnik and J. McKinnon, Phys. Rev. Lett. **61**, 141 (1988).
 - [2] P. Bizoń, Phys. Rev. Lett **64**, 2844 (1990).
 - [3] M. S. Volkov and D. V. Gal'tsov, Sov. J. Nucl. Phys. **51**, 1171 (1990); H. P. Kuenzle and A. K. M. Masood-ul-Alam, J. Math. Phys. **31**, 928 (1990).
 - [4] These hairy black-hole solutions are labeled by an integer parameter n which counts the number of nodes of the exterior Yang-Mills field.
 - [5] P. B. Yasskin, Phys. Rev. D **12**, 108 (1975).
 - [6] N. Straumann and Z.-H. Zhou, Phys. Lett. B **237**, 353 (1990); N. Straumann and Z.-H. Zhou, Phys. Lett. B **243**, 33 (1990).
 - [7] P. Bizoń and R. M. Wald, Phys. Lett. B **267**, 173 (1991).
 - [8] P. Breitenlohner, P. Forgács, and D. Maison, Nucl. Phys. B **383**, 357 (1992); P. Breitenlohner, P. Forgács, and D. Maison, Nucl. Phys. B **442**, 126 (1995).
 - [9] This fact refers to the case of a purely magnetic ansatz for the Yang-Mills field [7].
 - [10] V. Moncrief, Phys. Rev. D **9**, 2707 (1974); V. Moncrief, Phys. Rev. D **10**, 1057 (1974).
 - [11] S. Hod, Phys. Lett. B **713**, 505 (2012); S. Hod, Phys. Lett. B **718**, 1489 (2013) [arXiv:1304.6474].
 - [12] For an excellent review on the critical phenomena in gravitational collapse, see C. Gundlach and J. M. Martín-García, Living Rev. Relativity **10** (2007).
 - [13] M. W. Choptuik, T. Chmaj, and P. Bizoń, Phys. Rev. Lett. **77**, 424 (1996); C. Gundlach, Phys. Rev. D **55**, 6002 (1997).
 - [14] M. Choptuik, E. Hirschmann, and R. Marsa, Phys. Rev. D **60**, 124011 (1999).
 - [15] P. Bizoń and T. Chmaj, Phys. Rev. D **61**, 067501 (2000).
 - [16] O. Rinne, e-print arXiv:1409.6173.
 - [17] This fact refers to type I and type III critical behaviors, see [13–16] for details.
 - [18] Here $|p - p^*|$ is a measure for the distance of the initial data from the threshold (critical) solution [12].
 - [19] As emphasized above, the SU(2) Reissner-Nordström black-hole solution of the coupled Einstein-Yang-Mills equations is characterized by an *infinite* set of unstable perturbation modes [8]. Reference [16] provides, for the first time, numerical results for the first three eigenvalues.
 - [20] These are SU(2) Reissner-Nordström black holes whose magnetic charges are small on the scale set by the black-hole mass, see Eq. (10) below.
 - [21] We use natural units in which $G = c = \hbar = 1$.
 - [22] P. Bizoń, Phys. Lett B **259**, 53 (1991).
 - [23] Note that the near-horizon region $r \rightarrow r_+$ corresponds to $x \rightarrow -\infty$, whereas spatial infinity $r \rightarrow \infty$ corresponds to $x \rightarrow \infty$.
 - [24] Note that ω^2 plays the role of energy in the Schrödinger-like wave equation (11). Thus, eigenfrequencies with $\omega^2 < 0$ correspond to bound states of the system.
 - [25] J. A. Wheeler, Phys. Rev. **97**, 511 (1955); T. Regge and J. A. Wheeler, Phys. Rev. **108**, 1063 (1957).
 - [26] One can also take $\ell \equiv \ell_{\text{eff}} = \frac{-1-i\sqrt{3}}{2}$ which corresponds to stable modes (with $\Im\omega < 0$) that blow up at $x \rightarrow \pm\infty$.
 - [27] S. R. Dolan and A. C. Ottewill, Class. Quant. Grav. **26**, 225003 (2009).
 - [28] Note that the boundary conditions used in [27] for the analysis of the black-hole quasinormal resonances [that is, outgoing waves at spatial infinity: $\xi(x \rightarrow \infty) \sim e^{i\omega x}$, and ingoing waves at the black-hole horizon: $\xi(x \rightarrow -\infty) \sim e^{-i\omega x}$] correspond in our case [that is, for unstable (bound-state) modes with $\omega = i|\omega|$] to spatially bounded solutions with $\xi(x \rightarrow \infty) \sim e^{-|\omega|x} \rightarrow 0$ and $\xi(x \rightarrow -\infty) \sim e^{|\omega|x} \rightarrow 0$.
 - [29] Here we have used the values $\beta \equiv 1 - s^2 = 1 - 1 = 0$ ($s^2 = 1$ corresponds to electromagnetic perturbation fields) and $N \equiv n + 1/2 = 0 + 1/2 = 1/2$ ($n = 0$ corresponds to the fundamental eigenvalue) in equations (17)-(22) of [27].
 - [30] As discussed in [27], the validity of the expansion method (14) is restricted to the *fundamental* $n \lesssim l$ modes.
 - [31] This numerically computed [16] eigenvalue corresponds to a weakly-magnetized SU(2) Reissner-Nordström black hole with $r_+ = 10 \gg 1$. This horizon radius corresponds to a black-hole mass of $M = 5.05$ [see Eq. (5)]. These are the largest horizon-radius and black-hole mass studied numerically in [16].
 - [32] These modes correspond to the “excited energy states” of the system.
 - [33] Note that the eigenvalue (16) almost saturates the upper bound suggested in S. Hod, Phys. Lett. B **661**, 175 (2008) [arXiv:0803.0608].
 - [34] R. Fabbri, Phys. Rev. D **12**, 933 (1975).
 - [35] S. A. Teukolsky and W. H. Press, Astrophys. J. **193**, 443 (1974).
 - [36] Here we have substituted $a = 0$ (which corresponds to a

- non-rotating black hole), $\lambda = \ell(\ell + 1) - s(s + 1)$ (which is the angular eigenvalue corresponding to a non-rotating black hole), and $s = -1$ (which corresponds to electromagnetic perturbation fields) in Eq. (2.9) of [35] (known as the master radial Teukolsky equation). It is worth noting that our final result [see Eq. (40) below] is invariant under the transformation $s \rightarrow -s$.
- [37] S. Chandrasekhar, Proc. R. Soc. London **A 343**, 289 (1975).
- [38] J. M. Bardeen and W. H. Press, J. Math. Phys. **14**, 7 (1973).
- [39] See [40] for a similar low frequency matching procedure in the context of black-hole quasinormal resonances.
- [40] S. Hod, Phys. Lett. B **666**, 483 (2008) [arXiv:0810.5419]; S. Hod, Phys. Rev. D **88**, 084018 (2013) [arXiv:1311.3007].
- [41] D. N. Page, Phys. Rev. D **13**, 198 (1976).
- [42] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover Publications, New York, 1970).
- [43] Note that the overlap region $1 \ll z \ll 1/k$ exists in the small-frequency regime $k \ll 1$ [see Eqs. (19) and (23)].
- [44] Here we have used the relation $1 = e^{-i2\pi n}$ for $n = 1, 2, 3, \dots$. In addition, we have used here Eq. 6.1.23 of [42].
- [45] Curiously, we find the relation $\theta/2\phi = 1.0016$. One can therefore replace, with an accuracy of 0.05%, the term $\frac{4\theta-2\phi}{\sqrt{3}}$ in (37) and (39) by the simpler expression $\sqrt{3}\theta$. Taking cognizance of the expression (16) for the fundamental eigenvalue, which also contains an explicit factor of $\sqrt{3}$, one may suspect that the relation $\frac{4\theta-2\phi}{\sqrt{3}} \simeq \sqrt{3}\theta$ (with a remarkable accuracy of 0.05%) may be more than just a mere coincidence.
- [46] These numerically computed [16] eigenvalues correspond to a weakly-magnetized SU(2) Reissner-Nordström black hole with $r_+ = 10 \gg 1$. This horizon radius corresponds to a black-hole mass of $M = 5.05$ [see Eq. (5)]. These are the largest horizon-radius and black-hole mass studied numerically in [16].
- [47] It is worth emphasizing that the *zeroth*-order analytical formula (37) already provides a good description for the $n \geq 2$ unstable eigenvalues of the black hole. Specifically, one finds from (37) $M\omega_2^{(0)\text{ana}} = 3.66 \times 10^{-4}i$ which implies the fairly good agreement [see Eq. (42)] $\omega_2^{(0)\text{ana}}/\omega_2^{\text{num}} \simeq 1.025$ between the *zeroth*-order analytical formula (37) and the corresponding numerically computed [16] eigenvalue (42).
- [48] In the language of quantum mechanics, these eigenvalues correspond to excited bound-state energies.
- [49] As emphasized in [16], the SU(2) Reissner-Nordström black hole is only an *approximate* intermediate attractor because it has *infinitely* many unstable modes. It turns out that the coefficients of the higher ($n \geq 2$) unstable modes are small as compared to the coefficients of the first two unstable modes. Thus, the exponential growths of these higher unstable modes are almost invisible for a time period which is longer than the characteristic lifetime of the approximate intermediate attractor.